# $C^*$-algebras associated to Temperley-Lieb polynomials

(Joint work with Sergey Neshveyev)

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<u>Plan:</u>

TL-polynomials $\xrightarrow{(1)}$ subproduct systems

(3) ↓                                  ↓ (2)

Compact quantum groups $\overset{}{\underset{(4)}{\longleftrightarrow}}$ $C^*$-algebras

# Subproduct systems

A subproduct system consists of

- a family of Hilbert spaces $\mathcal{H} = (H_n)_{n=0}^{\infty}$
- isometries $V_{k,l} : H_{k+l} \longrightarrow H_k \otimes H_l$, $\quad k, l \in \mathbb{Z}_+$

such that

1) $\dim H_0 = 1$, $\dim H_1 < \infty$

2)
$$\begin{array}{ccc}
H_{k+l+s} & \xrightarrow{\;V_{k+l,s}\;} & H_{k+l} \otimes H_s \\
{\scriptstyle V_{k,l+s}} \big\downarrow & \circlearrowleft & \big\downarrow {\scriptstyle V_{k,l} \otimes 1} \\
H_k \otimes H_{l+s} & \xrightarrow[\;1 \otimes V_{l,s}\;]{} & H_k \otimes H_l \otimes H_s
\end{array}$$

Let $\mathcal{H} = (H_n)_{n=0}^{\infty}$ be a subproduct system.

- Fock space: $\mathcal{F}_{\mathcal{H}} = \bigoplus_{n=0}^{\infty} H_n$

- "Creation operators": $S_\xi : \mathcal{F}_{\mathcal{H}} \to \mathcal{F}_{\mathcal{H}}$ , $\xi \in H_1$

$$S_\xi(\zeta) = V_{n,1}^*(\xi \otimes \zeta) \quad , \quad \zeta \in H_n$$

$$H_n \to H_{n+1}$$

- Toeplitz algebra: $\underline{\mathcal{T}_{\mathcal{H}}} = C^*(1, S_1, S_2, \ldots, S_n)$
where $S_i = S_{\xi_i}$ for an o.n.b. $(\xi_i)_i$ in $H_1$.

- Cuntz-Pimsner algebra: $\underline{\mathcal{O}_{\mathcal{H}}} = \mathcal{T}_{\mathcal{H}} / K(\mathcal{F}_{\mathcal{H}})$

$$\left( \sum_i S_i S_i^* = 1 - e_0 \right)$$

## Main example: $\mathcal{H}_P$

- $H$: Hilbert space, $\dim H = m < \infty$
- Fix $P \in H \otimes H$, $P \neq 0$
- Let $e: H \otimes H \to \mathbb{C}P$ be the projection
- Define $f_0 = 1 \in B(\mathbb{C})$, $f_1 = 1 \in B(H)$ and

$$n \geq 2 \quad f_n = 1 - \bigvee_{k=0}^{n-2} 1^{\otimes k} \otimes e \otimes 1^{\otimes(n-k-2)} \in B(H^{\otimes n})$$

- $H_n := f_n H^{\otimes n}$ $\quad H^{\otimes(k+\ell)}$

Then $H_{k+\ell} \subseteq H_k \otimes H_\ell$ defines a (standard) subproduct system $\mathcal{H}_P$.

Example

- Let $\{\xi_1, \xi_2\}$ be the standard basis in $\overset{H}{\underset{\|}{}} \mathbb{C}^2$
- $p = \xi_1 \otimes \xi_2 - \xi_2 \otimes \xi_1 \in \mathbb{C}^2 \otimes \mathbb{C}^2$
- Then $H_n = \mathrm{Sym}^n(\mathbb{C}^2)$

Arveson: $\mathcal{O}_p \cong C(S^3)$

"Proof":

- $\mathcal{O}_p$ is abelian with $\mathrm{spec}\, \mathcal{O}_p \subseteq S^3$

$\rightsquigarrow$ Surjective $*$-hom $\varphi : C(S^3) \to \mathcal{O}_p$

- $\mathcal{U}(2) \curvearrowright \mathcal{O}_p$, and $\varphi$ is equivariant
- The action $\mathcal{U}(2) \curvearrowright S^3$ is transitive

# Temperley-Lieb polynomials

**Def.** $p \in H \otimes H$ is Temperley-Lieb if the projection $e : H \otimes H \to \mathbb{C}p$ satisfies

$$(e \otimes 1)(1 \otimes e)(e \otimes 1) = \frac{1}{\lambda}(e \otimes 1), \quad \lambda > 0.$$

**Remark:**

- $TL_n(\lambda^{-1}) \cong C^*(1^{\otimes k} \otimes e \otimes 1^{\otimes(n-k-2)} \mid 0 \leq k \leq n-2) \subseteq B(H^{\otimes n})$

⇝ The projections $p_n$ defining $\mathcal{H}_p$ are the "Jones-Wenzl projections".

**Goal:** Understand $\mathcal{H}_p$ (and $\mathcal{J}_p$, $\mathcal{O}_p$) where $p$ is a Temperley-Lieb.

## Step 1: Relations in $\mathcal{T}_P$

- $c = C(\mathbb{Z}_+ \cup \{\infty\}) \xrightarrow{i} \mathcal{T}_P$, $\quad i(x)\xi = x(n)\xi$ for $\xi \in H_n$
- Let $\gamma : c \to c$ denote the left shift

**Prop.** Assume $P = \sum_{i,j} a_{ij}\, \xi_i \otimes \xi_j$ is Temperley-Lieb.
Let $q \in (0,1]$ be such that $\mathrm{Tr}(A^*A) = q + q^{-1}$, $A = (a_{ij})_{i,j}$.
The following relations hold in $\mathcal{T}_P$:

$$f S_i = S_i \gamma(f), \qquad \sum_i S_i S_i^* = 1 - e_0, \qquad \sum_{i,j} a_{ij} S_i S_j = 0$$

$$S_i^* S_j + \varphi \sum_{k,l=1}^{m} a_{ik} \bar{a}_{jl} S_k S_l^* = \delta_{ij} 1$$

where $\varphi \in c$ is given by $\varphi(n) = \dfrac{[n]_q}{[n+1]_q}$.

$$\varphi(n) \to q$$

<u>Idea</u>: Use equivariance to study $\mathcal{H}_p$ ($\mathcal{T}_p$ and $\mathcal{O}_p$).

Assume $G$ is a compact quantum group, and let $\mathcal{H} = (H_n)_{n=0}^{\infty}$ be a subproduct system.

$\mathcal{H}$ is <u>$G$-equivariant</u> if

- there are unitary $G$-representations $U_n$ on $H_n$, and
- the isometries $V_{k,l}$ are intertwiners.

In this situation $B(\mathcal{F}_{\mathcal{H}}) \curvearrowleft G$ via

$$U = \bigoplus_{n=0}^{\infty} U_n : \quad T \mapsto U(T \otimes 1) U^*.$$

$\rightsquigarrow$ $K(\mathcal{F}_{\mathcal{H}})$, $\mathcal{T}_{\mathcal{H}}$, $\mathcal{O}_{\mathcal{H}}$ $\curvearrowleft G$.

Step 2: Find a "nice symmetry group".

Observation: $\mathcal{H}_P$ is $G$-equivariant if there are representations $V$ on $H_1$ and $d$ on $\mathbb{C}$ s.t.

$$(V \otimes V)(P \otimes 1) = P \otimes d \quad \text{in } H \otimes H \otimes \mathbb{C}[G].$$

Remark:

- There can be many such $G$
- $d = 1$ could also work

$e : H \otimes H \to \mathbb{C}P$

$\cap$

$\mathrm{Mor}_G(V \otimes V, d)$

$f_n \in \mathrm{End}(V^{\otimes n})$

$V_n \subseteq V^{\otimes n}$

Example.

- $P = \xi_1 \otimes \xi_2 - \xi_2 \otimes \xi_1 \in \mathbb{C}^2 \otimes \mathbb{C}^2$
- $\mathcal{U}(2) \curvearrowright \mathbb{C}^2$
- $V \in \mathcal{U}(2) \rightsquigarrow VP = \det(V)\,P$.

**Def** (Mrozinski) For $A \in GL_m\mathbb{C}$ define $\mathbb{C}[\tilde{O}^+_A]$ as the universal unital $*$-algebra generated by $d$, $v_{ij}$, $1 \le i,j \le m$ s.t.

- $V = (v_{ij})_{i,j}$ and $d$ are unitaries, and
- $VAV^t = dA$.

$\mathbb{C}[\tilde{O}^+_A]$ is a Hopf $*$-algebra with

$$\Delta(d) = d \otimes d, \quad \Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj}$$

**Remark.**

- If $P = \sum_i \xi_i \otimes A\xi_i$, then $(V \otimes V)(P \otimes 1) = P \otimes d$.

## Example:

- For $q \in (0,1]$ and $A_q = \begin{pmatrix} 0 & q^{-1/2} \\ -q^{1/2} & 0 \end{pmatrix}$ we have:

$$\tilde{O}^+_{A_q} = U_q(2)$$

- Here $P = \sum_i \xi_i \otimes A\xi_i = q^{-1/2} \xi_1 \otimes \xi_2 - q^{1/2} \xi_2 \otimes \xi_1$

Prop: Let $A \in GL_m\mathbb{C}$, and put $P = \sum_i \xi_i \otimes A\xi_i$.

Then TFAE:

(i) $V \in B(\mathbb{C}^m) \otimes \mathbb{C}[\tilde{O}^+_A]$ is irreducible

(ii) $A\bar{A}$ is unitary up to a scalar

(iii) $P$ is Temperley-Lieb

In this case $\tilde{O}_A^+$ is a "$U(2)$-deformation":

$$R[\tilde{O}_A^+] \cong R[U(2)]$$

<u>Def.</u> For $q \in (0,1]$, $m \geq 2$, define the set

$$\mathcal{M}_q^m = \{ A \in GL_m\mathbb{C} \mid A\bar{A} \text{ unitary}, \ Tr(A^*A) = q + q^{-1} \}.$$

<u>Def (Mrozinski).</u> Let $q \in (0,1]$. Fix $A \in \mathcal{M}_q^m$, $C \in \mathcal{M}_q^k$.
$B(A,C)$ is the universal unital algebra generated by $z, z^{-1}, y_{ij}$, $1 \leq i \leq k$, $1 \leq j \leq m$ such that

$$YAY^t = zC, \quad Y^t \bar{C} Y = z\bar{A}, \quad zz^{-1} = z^{-1}z = 1.$$

$$Y = (y_{ij})_{i,j}$$

**Thm (Mrozinski).** Let $A \in \mathcal{M}^m_q$, $C \in \mathcal{M}^k_q$. Then

$B(A,C)$ is a $\mathbb{C}\{\tilde{O}^+_A\}$-$\mathbb{C}\{\tilde{O}^+_C\}$-Galois object:

$$\mathbb{C}\{\tilde{O}^+_C\} \otimes B(A,C) \xleftarrow{\delta_C} B(A,C) \xrightarrow{\delta_A} B(A,C) \otimes \mathbb{C}\{\tilde{O}^+_A\}$$

$$(z \otimes \delta_A)(Y) = Y_{12} V^A_{13}, \quad \delta_A(z) = z \otimes d^A$$

$$(z \otimes \delta_C)(Y) = V^C_{12} Y_{13}, \quad \delta_C(z) = d^C \otimes z$$

**Lemma.** $B(A,C)$ is a $*$-algebra with

$$z^* = z^{-1}, \quad Y^c = C^t Y (A^t)^{-1} z.$$

**Prop.** A $C^*$-envelope $\tilde{B}(A,C)$ of $B(A,C)$ exists and defines a $C^*$-algebraic $\tilde{O}^+_A$-$\tilde{O}^+_C$-Galois object.

In particular: $\rightsquigarrow U_q(2)$

- $q \in (0,1]$, $A_q = \begin{pmatrix} 0 & q^{-\frac{1}{2}} \\ -q^{\frac{1}{2}} & 0 \end{pmatrix} \in \mathcal{M}_q^2$
- $A \in \mathcal{M}_q^m \rightsquigarrow A = \begin{pmatrix} 0 & & a_1 \\ & a_{m-1} \cdot^{\cdot^{\cdot}} & \\ a_m & & 0 \end{pmatrix}$, $|a_i a_{m-i+1}| = 1$

Lemma. $\tilde{B}(A_q, A)$ is the universal $C^*$-algebra generated by $y_1, y_2, \dots, y_m$ and $z$ s.t.

1) $\begin{pmatrix} q^{1/2}\bar{a}_1 y_m^* & q^{1/2}\bar{a}_2 y_{m-1}^* & \cdots & q^{1/2}\bar{a}_m y_1^* \\ y_1 & y_2 & \cdots & y_m \end{pmatrix}$ is unitary

2) $z y_i z^* = -a_i \bar{a}_{m-i+1} y_i$

Theorem. $\mathcal{O}_P \cong C^*(y_1, y_2, \dots, y_m) \subseteq \tilde{B}(A_q, A)$.

- $u = \bigoplus_{n=0}^{\infty} (-A\bar{A})^{\otimes n} \in B(\mathcal{F}_p)$

- $\beta = \mathrm{Ad}\, u : \mathcal{J}_p \to \mathcal{J}_p \rightsquigarrow \beta(S_i) = -a_i \bar{a}_{m-i+1} S_i$

<u>Cor.</u> $\tilde{B}(A_q, A) \cong \mathcal{O}_p \rtimes_{\bar{\beta}} \mathbb{Z}$

<u>Prop.</u> $\mathcal{J}_p$ is a universal $C^*$-algebra generated by $c$ and elements $S_1, S_2, \ldots, S_m$ with the relations given earlier.

pecial case:

- $A\bar{A} = \pm 1$, $Tr(A^*A) = q + q^{-1}$

$\rightsquigarrow \quad \mathcal{O}_P \cong B(SU_q(2), O_A^+)$

- $P = q^{-1/2}\xi_1 \otimes \xi_2 - q^{1/2}\xi_2 \otimes \xi_1$, $A = \begin{pmatrix} 0 & q^{-1/2} \\ -q^{1/2} & 0 \end{pmatrix}$

$\rightsquigarrow \quad \underline{\mathcal{O}_P \cong C(SU_q(2))}$

$SU(2) \cong S^3$

# K-theory

- $i : \mathbb{C} \to \mathcal{J}_p$ induces an isomorphism in $KK^{\tilde{O}_A^+}$

- Proof
  - "BC for $U_q(2)$"
  - $KK^{U_q(2)} \cong KK^{\tilde{O}_A^+}$

$$0 \to K \to \mathcal{J}_p \to \mathcal{O}_p \to 0$$

- Find (Arici-Kaad) inverse $i : \mathbb{C} \to \mathcal{J}_p$

$$K_0(\mathcal{O}_p) = \mathbb{Z}/(m-2)\mathbb{Z}$$

$$K_1(\mathcal{O}_p) = \begin{cases} \mathbb{Z} & : m = 2 \\ 0 & : m \geq 3 \end{cases}$$